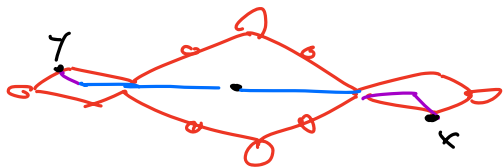


Topological Model for Superattracting

$f(z) = z^2 + c$, superattracting per. $u \mapsto K(f)$ loc. conv.

Legal arcs: $[x, y]$ for $x \neq y$

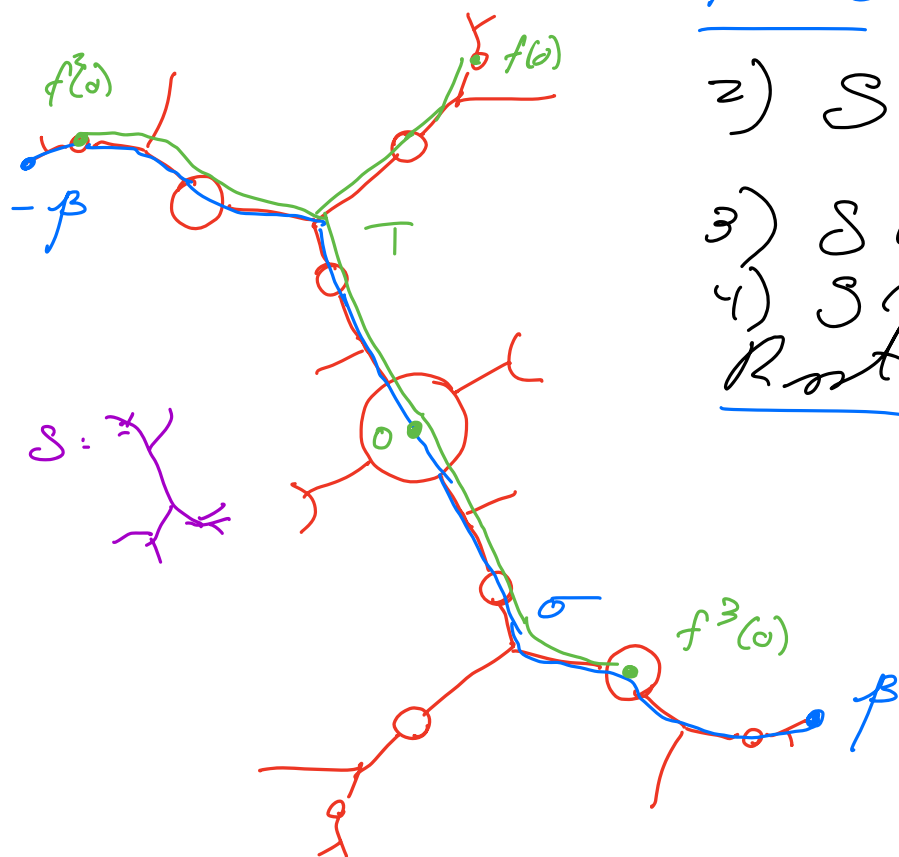


Def. $\beta := \gamma(0)$, landing point of R_0

Spine $\sigma = [\beta, -\beta]$, Skeleton $S = \bigcup_{u=0}^{\infty} f^{-u}(\sigma)$

Hubbard tree $T = \bigcup_{u, v} [f^u(0), f^v(0)]$

Ex: Misiurewicz



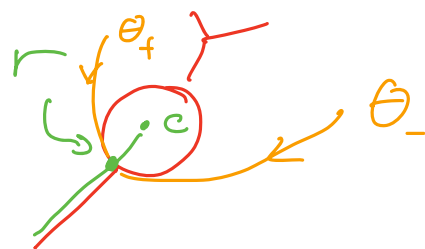
Facts 1) c is a tip of T

2) $S = \bigcup_{u=0}^{\infty} f^{-u}(T)$

3) $S \cap T = \{\text{cut pts}\}$

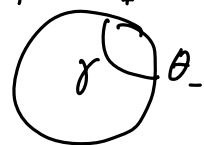
4) $S \cap A(0)$ is dense in S

Rest $r \in \partial A_+(c) \cap T$



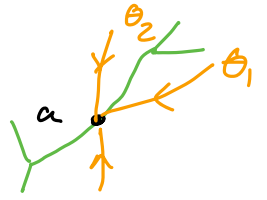
γ is the geodesic in $\mathbb{D}$ between θ_- , θ_+

$E: S' \rightarrow S'$
 $x \mapsto 2x$

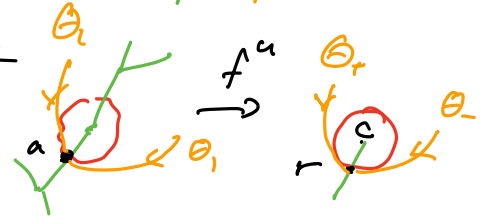


Theorem. $Z = \overline{\bigcup_{u \geq 0} E^{-u}(\gamma)}$ is the limit set of external rays for $K(f)$

Proof. 1) $a \in T \cap J$, $\theta_1 \neq \theta_2$ bound the smallest interval in $S' \setminus \gamma^{-1}(a)$

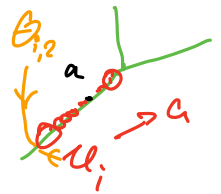


1.1) $a \in \partial U$, $U \subset A(0)$. Pass to $f^u(u) = A_0(c)$, $f^u(a) \in \partial A_0(c) \cap T$



Since c is a tip, $f^u(a) = r$, $E^u(\theta_2) = \theta_+$, $E^u(\theta_1) = \theta_-$.
 $\Rightarrow (\theta_1, \theta_2)$ is a preimage of γ

1.2) a is the tail of component U_i of $A(0)$



Let $\theta_{i,1} \neq \theta_{i,2}$ be the angles for part 1.1 for points on $\partial U_i \cap T$; $(\theta_{i,1}, \theta_{i,2}) \in Z$

$\theta_{i,1}$ increasing, $\theta_{i,2}$ decreasing, $\theta_{i,1} \rightarrow \theta_1$, $\theta_{i,2} \rightarrow \theta_2$
 land at $a \Leftrightarrow (\theta_1, \theta_2) \in Z$

2) Since $T \cap A(0)$ is dense in T , the same is true for all $T \cap J$.

3) Since $S = \bigcup_{u \geq 0} f^{-u}(T)$, the same is true for all cut points.

Mandelbrot Set

$$f_c(z) = z^2 + c$$

$$\mathcal{M} = \{c \in \mathbb{C} \mid J(f_c), K(f_c) \text{ are connected}\}$$

$$= \{c \in \mathbb{C} \mid \{f_c^n(0)\} \text{ is bounded}\}$$

Prop. $\mathcal{M} \subset \mathbb{C}$ is compact and $\mathbb{C} \setminus \mathcal{M}$ is connected

Proof. 1) $\mathcal{M} \subset \overline{\mathbb{D}_2}$:

$$\begin{aligned} \cdot |z| \geq |c| > 2 : |f_c(z)| &= |z^2 + c| \geq |z|^2 - |c| \\ &\geq |z| \cdot |c| - |z| \\ &= |z| \cdot (|c| - 1) \end{aligned}$$

$$\Rightarrow |f_c^{n+1}(0)| = |f_c^n(c)| \rightarrow \infty \Rightarrow c \notin \mathcal{M}$$

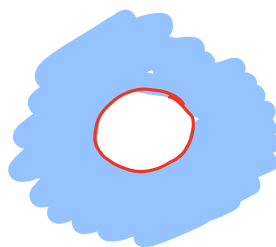
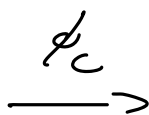
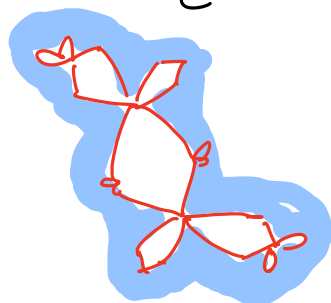
$$2) \mathcal{M} = \bigcap_{n=1}^{\infty} \{c \in \mathbb{C} \mid |f_c^n(0)| \leq 2\} \text{ is compact}$$

$$3) \mathbb{C} \setminus \mathcal{M} \text{ connected by any path:}$$

• Böttcher map: $\phi_c(z) = \lim_{n \rightarrow \infty} (f_c^n(z))^{1/2^n} \setminus \{f_c^n(0) > 2\}$

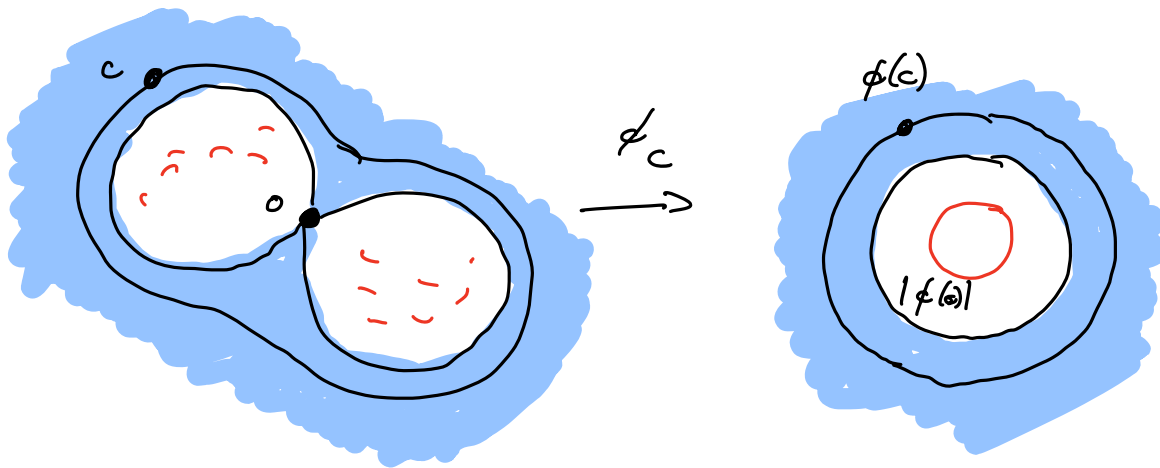
$$= \lim_{n \rightarrow \infty} z \prod_{k=1}^n \left(1 + \frac{c}{f_c^k(z)}\right)^{1/2^k}$$

$$1) c \in \mathcal{M}: \phi_c: \hat{\mathbb{C}} \setminus K(f_c) \xrightarrow{\sim} \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$$

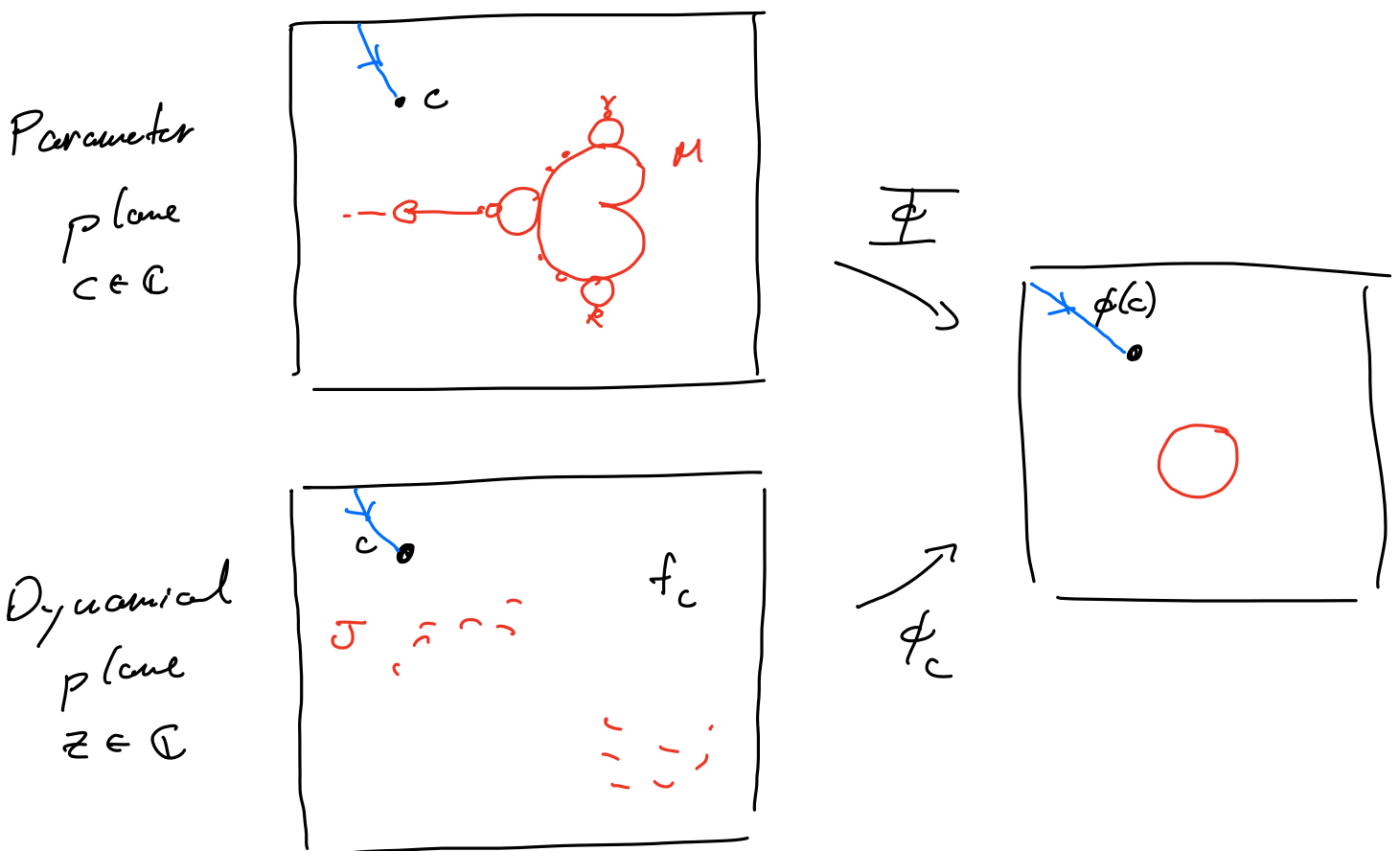


$$2) c \notin M : \phi_c : \hat{\mathbb{C}} \setminus \{|\phi| \leq |\phi(c)|\} \xrightarrow{\sim} \hat{\mathbb{C}} \setminus \overline{D}_r$$

$r > 1$



Thm. The function $\Phi(c) = \phi_c(c)$ defines an isomorphism $\Phi: \mathbb{C} \setminus M \xrightarrow{\sim} \mathbb{C} \setminus \overline{D}$. In particular, M is connected.



Proof. 1) Φ is bijective ✓

2) $\mathbb{F}$ is proper:

$$G_c(z) = \log |f_c(z)| = \log |z| + \sum_{n=1}^{\infty} \frac{1}{2^n} \log \left| 1 + \frac{c}{f_c^n(z)^2} \right|$$

$$\cdot |z| \geq |c| > 2 \Rightarrow |f_c(z)| > |c| \Leftrightarrow \left| 1 + \frac{c}{f_c^n(z)^2} \right| \leq 2$$

$$\cdot \log |\mathbb{F}(c)| - \log |c| \text{ is bounded } \forall c$$

$\Rightarrow \mathbb{F}$ is proper

3) $\mathbb{F}$ has degree 1:

$\left| \frac{\mathbb{F}(c)}{c} \right|$ is bounded $\Rightarrow \mathbb{F}(c)$ has a single pole at ∞ . Any principle $\Leftrightarrow$ degree 1

Conj (MLC) $\mathcal{M}$ is locally connected

Def. $E \subset \hat{\mathbb{C}}$, $\Lambda \subset \mathbb{C}$ domain, $\lambda_0 \in \Lambda$.

A holomorphic motion of E parametrized by Λ is a function $\phi: \Lambda \times E \rightarrow \hat{\mathbb{C}}$,
 $\phi_\lambda(z) := \phi(\lambda, z)$ s.t.

1) For each $z \in E$, $(\lambda \mapsto \phi_\lambda(z))$ is hol. on Λ ,

2) For each $\lambda \in \Lambda$, $(z \mapsto \phi_\lambda(z))$ is injective

3) $\phi(\lambda_0, z) = z$

Lemma. (λ -lemma) If $\phi: \Lambda \times E \rightarrow \hat{\mathbb{C}}$ is a hol. motion,

then f extends to a continuous function $\bar{f}: \Lambda \times \bar{E} \rightarrow \hat{C}$
 which is a lab. motor of $\bar{E}/\Lambda$.
 $\hookrightarrow$ Proof as HW

Def. Family of lab. maps $f: \Lambda \times \hat{C} \rightarrow \hat{C}$ lab
 $f_\lambda(z) = f(\lambda, z)$

Julia set varies holomorphically if $\exists$ a
 lab. motor $\phi: \Lambda \times J(f_{\lambda_0}) \rightarrow \hat{C}$ s.t.

$$\begin{array}{ccc} J(f_{\lambda_0}) & \xrightarrow{f_{\lambda_0}} & J(f_{\lambda_0}) \\ \phi_{\lambda_0} \downarrow & & \downarrow \phi_{\lambda_0} \\ J(f_\lambda) & \xrightarrow{f_\lambda} & J(f_\lambda) \end{array} \quad f_\lambda(\phi_\lambda(z)) = \phi_\lambda(f_{\lambda_0}(z)).$$

Thm/Def. Family f_λ is stable if the following
 equivalences hold:

- 1) Julia set of f_λ varies holomorphically
- 2) # of attracting cycles of f_λ is locally const.
- 3) Every attracting, indifferent, or repelling per.
 pt of f_{λ_0} remains attr., indiff., or repelling for
 f_λ for λ near λ_0 .
- 4) If $\{c_i(\lambda)\}_i$ are the crit. points of f_λ ,
 then $\{\lambda \mapsto f_\lambda^n(c_i(\lambda))\}_n$ is a normal family.

5) $c_i(\lambda) \in J(f_\lambda)$ iff $c_i(\lambda_0) \in J(f_{\lambda_0})$

Def. f_λ / Λ family.

stable locus

$$\Lambda_{stab} = \{ \lambda_0 \in \Lambda \mid f_\lambda \text{ is stable in a neighborhood of } \lambda_0 \}$$

hyperbolic locus

$$\Lambda_{hyp} = \{ \lambda \in \Lambda \mid f_\lambda \text{ is hyperbolic} \}.$$

Thm.

$$\Lambda_{hyp} \subset \Lambda_{stab} \subset \Lambda$$

$\begin{matrix} \text{open} & \text{open} \\ \text{+ closed} & \text{+ dense} \end{matrix}$

Proof. 1) $\Lambda_{stab} \subset \Lambda$:

$N(\lambda) = \#$ of attracting cycles of f_λ

Attr. cycles persist, so if $\lambda_n \rightarrow \lambda$,

$$\limsup_{n \rightarrow \infty} N(\lambda_n) \geq N(\lambda).$$

• $\{ N(\lambda) \text{ is loc. const.} \} \subset \Lambda$ is open since f_λ stable (2)

2) f_{λ_0} hyp: crit pts $\rightarrow$ attr. cycles, so $c_i(\lambda) \notin J(f_{\lambda_0})$.

• For λ near λ_0 , attr. cycles remain attr, so f_λ is hyp. $\Rightarrow \Lambda_{hyp}$ is open and also $c_i(\lambda) \notin J(f_\lambda)$,
so $\Lambda_{hyp} \subset \Lambda_{stab}$. (5)

3) f_{λ_0} is not hyp, $\lambda_0 \in \Lambda_{stab}$:

$\exists c_i(\lambda_0)$ st. $f_{\lambda_0}^{-n}(c_i(\lambda_0)) \rightarrow$ atr. cycle.

a) $c_i(\lambda_0) \in J(f_{\lambda_0})$: $J(f_{\lambda})$ moves back, so
 $c_i(\lambda) \in J(f_{\lambda})$ for nearby $\lambda \in \Lambda_{stab}$.

b) $c_i(\lambda_0) \in F(f_{\lambda_0})$: $c_i(\lambda_0) \rightarrow$ parabolic cycle

(3) $\Rightarrow$ parabolic cycle persists for f_{λ} and attracts $c_i(\lambda)$

$\Rightarrow f_{\lambda}$ is not hyp for nearby $\lambda \Rightarrow \Lambda_{hyp}$ is closed

Prop. 1) f_c is stable / $\mathbb{C} \setminus \mathcal{M}$

2) $\mathcal{M}_{hyp} \subset \mathcal{M}_{stab} = \mathcal{M}^0$, $\partial \mathcal{M}$ is where f_c is not stable

3) $\mathcal{M}_{hyp}$ is the union of some components of $\mathcal{M}^0$

Proof.

$c_0 \in \mathcal{M}^0$: $|f_c^{-n}(c_0)| \leq 2 \Rightarrow \{c \mapsto f_c^{-n}(c)\}_n$ is normal

(4) $\Rightarrow \mathcal{M}^0 \subset \mathcal{M}_{stab}$

$c_0 \notin \mathcal{M}$: $f_c^{-n}(c_0) \rightarrow \infty$:

$c_0 \in \partial \mathcal{M}$: $\{c \mapsto f_c^{-n}(c)\}$ can't be normal near c_0
 $\mathcal{M}_{stab} \subset \mathcal{M}^0$

Conj. (DH) $\mathcal{M}_{hyp} = \mathcal{M}^0$

Thm. $\mathcal{M}LC \Rightarrow DH$